The logical structure of relativity theory

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We discuss our Fulbright work in Budapest on applying the techniques and the spirit of modern mathematical logic to relativity theory, and lessons learned from having lived and studied in Hungary.

1. Introduction

My Fulbright project rests on the hypothesis that the theory of relativity may be profitably investigated using the techniques of modern mathematical logic. The theory of relativity is regarded as one of the fundamental contributions to human thought; the theoretical and empirical implications of this theory have shaped the physical, mathematical, and philosophical views of the world in which we live. It is worthwhile, therefore, to investigate the foundations of relativity in more detail. On what assumptions does relativity rest, what are the reasons behind its predictions, and how do relativistic hypothesis figure in relativistic arguments? With the help of my advisors, for the past eight months I have investigated relativity as a formal mathematical theory and have applied the tools of mathematical logic to it. As a result I have learned much about relativity, mathematics in general, and logic in particular. In this paper, I discuss the theories that I have studied and the problems that I have considered. I begin by discussing the character of modern logic and its role in my project. I then give an example of an elementary relativity theory and then discuss some of its properties. Finally, I talk about my current work, and conclude with some personal remarks on what I have gained from my studies in Hungary.

2. The spirit of modern logic

Before discussing the results of my investigations, it is worthwhile to study the logical background that underlies the endeavor.

2.1. The origin of logic

Logic is reasoning about reasoning; it studies why some arguments are valid while others are not, what constitutes sound deduction, and how the truth of a statement relates to the truth of others. Classical logic was focused on such arguments as "Every Greek is mortal, Socrates is a Greek, therefore Socrates is mortal". The classification of these inferences was completed long ago (by Aristotle), and for a long time logic was regarded as a branch of rhetoric and philosophy. As the discipline matured, though, logic became a branch of mathematics. Modern logic offers precise definitions for the concepts it treats and it insists upon rigorous proof as the means of persuasion. Although mathematical style pervades the logic of today and can both clarify and obscure formal investigations, the object of modern logic is still human reasoning. Modern logic formalizes the notions of theory, validity, and proof, promoting them from fluid intuitions to mathematical concepts. Indeed, mathematical logic provides a definition for the concept of definition!

2.2. Logicization

To specify a logical theory, we must choose a formal language and use it to express the basic assumptions we wish to make about a chosen domain of discourse. Once the theory has been specified, a number of interesting questions may be treated. We may ask, for example, which statements are provable, whether our theory is consistent, and the extent to which the truth of one statement depends upon the truth of another. To help answer such questions, the tools of mathematical logic may be employed: mathematicians have discovered powerful theorems that give conditions on when formulas are provable, when a theory is complete (all true formulas of our domain are provable in the theory) or incomplete (some true formulas must be unprovable), and whether we may mechanically deduce in a finite number of steps whether any given statement is true. After capturing the basic assumptions of our domain, we investigate the formal theory we have obtained. We may discover, for example, that some statement which we would expect to be consequences of our formal theory are, in fact, not provable. Conversely, we may find proofs for some statements which we expect to be false. In the course of our investigations we may also find that the approach that underlies our choice of assumptions ought to be changed entirely. In each of these situations we modify of the axioms; we reconceive the basic truths of our chosen domain. The result is a new formal theory, and by repeating this process we obtain a hierarchy of formal theories, each expressing its own point of view on our subject.

2.3. Prospects of logicization for relativity

There does not appear to be any *a priori* restriction on the domains of discourse to which the logical method can give insight. Indeed, logic has been applied to economics, linguistics, biology, and many others. The goal of my research is to study the extent to which the logical method may be successfully applied to relativity theory. The language of relativity theory is quite simple: it talks only about light, observers, events, and a handful of a few other concepts. It is therefore reasonable to suppose that relativity may be formalized and that the resulting formal theory is manageable. That is, we may hope that formal theory of relativity will be rich enough to admit interesting theorems, but not too rich to make those theorems impossible to discover and to prove. I am pleased to report that these expectations have been borne out and that the formalization of relativity theory has been successful so far.

3. A special relativity theory

3.1. Basic axioms

For the first few months of my project, I studied the various relativity theories that my advisors have proposed. In this section, I discuss a particular formal 3 theory, called BASAX (for "BASic AXioms"), which is supposed to capture some assumptions of special relativity. The design of a theory of relativity in general, and BASAX in particular, begins with a study of the language employed in relativity. Nouns such as *observer*, and *photon*, and verbs such as *observe*, are isolated. Then, using these terms, assumptions such as "nothing travels faster than light" and "no observer is special" are formalized and regarded as axioms. The nouns of BASAX are *trace*, *observer*, *body*, *photon*, *line*, and *event*. We shall need only one verb, *observe*, and three adjectives, *at rest*, *fast*, and *slow*. Intuitively, observers observe events, which are made up of bodies, and traces, which are the paths that bodies mark out in time. Fast lines are the traces of photons (which play a distinguished role in relativity), and slow lines are the traces of things that move slower than light. In English, the axioms of BASAX assert:

1. [Observers] and [photons] are [bodies].
2. With respect to each [observer], the [trace] of each [body] is a [line].
3. Each [observer] [observes] itself [at rest].
4. Each [slow line] is the [trace] of an [observer] with respect to some [observer].
5. Each [fast line] is the [trace] of a [photon] with respect to some [observer].
6. All [observers] [observe] the same set of [events].
7. Each [observer] [observes] the [trace] of each [photon] as a [fast line].

Axiom 1 is straightforward: it puts under the common heading (*body*) two classes of objects that we wish to consider: *observer* and *photon*.

Axiom 2 is somewhat ad hoc: it implies that bodies move at a uniform velocity, that is, they do not accelerate. This assumption simplifies dealing with the theory. Later, we will drop this axiom and allow for accelerating observers, but with an added cost of making the theory mathematically more complex.

Axiom 3 is called the Galileian principle of relativity. It states that an observer can, without committing any impropriety, regard himself at rest and all others as in motion with respect to him.

Axiom 4 may be regarded as a license for performing thought experiments. When we are investigating some phenomenon geometrically, axiom 4 guarantees that each slow line that we draw will be the trace of *some* body.

Axiom 5 is similar; it says that each fast line will represent *some* photon.

Axiom 6 asserts that the speed of light is the same for all observers, an assumption that plays a special role in relativity theory. Like axiom 2, this axiom is somewhat restrictive. Indeed, it is not entirely true: the velocity of light in water is not the same as the velocity of light in a vacuum. By adopting this axiom, we were strict ourselves to situations in which light moves through a uniform medium.

3.2. Elementary consequences

The six axioms of BASAX constitute a simple theory of relativity. As we shall see in the next section, even though there are only a handful of assumptions, many interesting results follow from them. One such result states that no observer can travel at the speed of light. In other words, photons are distinguished. This result squares with what has been predicted in the past by physicists, but it is interesting to note that our result has been obtained in a different manner. Instead of appealing to an equation, our result has been derived purely logically. Moreover, the concepts of mass and energy play no role in our axioms, but the usual proof of this result *does* make use of these concepts. Put another way, we can say that it is already true in relativistic kinematics that no observer can travel at the speed of light; we do not need relativistic mechanics (which makes use of mass and energy) to infer this. Another interesting result gives the condition on which an observer can travel faster than light. It turns out that in three-dimensional space, like the space in which we live, faster-than-light travel cannot occur, although it is possible under special conditions.1 Like the previous result, this theorem makes no use of the 1It turns out that if the dimension of space is assumed to be one, that is, if bodies travel along a straight line, then faster-than-light travel is possible. 5 concept of mass, which is involved in the usual arguments excluding the possibility of faster-than-light travel. Finally, there is a famous prediction of special relativity which is a consequence of the basic axioms. It turns out that the concept of simultaneity cannot be independent of observers; it is possible that what is simultaneous for one observer is not simultaneous to another. My own work in the autumn was a study of the axioms which give rise to this "relativity of simultaneity" and a related phenomenon in which observers can disagree about the order in which two events occurred.

3.3. Modifying the theory

The logical study of relativity theory does not end with the specification and investigation of a single theory like BASAX. In order to understand more fully what relativistic assumptions imply what relativistic predictions, other, competing theories are proposed. The basic axioms have been weakened, strengthened, abandoned, each time yielding a new theory that, in a sense, has its own point of view. In their seven years of work on this project, my advisors have proposed more than four dozen special relativity theories. The result has been a penetrating investigation of relativity by studying the consequences of making small changes to the axioms.

4. Generalization and outlook

4.1. From special to general relativity

So far, we have focused only on special relativity theory. In special relativity, a central hypothesis is that no acceleration takes place. This assumption was captured in axiom 2 of BASAX, and it arises in one form or another in each of the special relativity theories that my advisors have proposed.

Even though it has practical benefits of simplifying calculations, we may regard the assumption that bodies do not accelerate as an *ad hoc* device, and we should modify this assumption if we want to make more accurate models of our experience in which bodies move at non-uniform velocities. Einstein thought this way, too, and proposed a way to get rid of it; the result is called general relativity theory.

The special principle of relativity says that if two observers move in straight lines with respect to each other, then they are the same from a physical point of view; the laws of physics according to one will be essentially the same as for the other. The general principle of relativity is a straightforward generalization of the special principle. The general principle says that the laws of physics will be essentially the same for all observers *no matter what their relative motion may be*. Rather than trying to modify the hypothesis that no acceleration takes place, Einstein's solution was to simply drop it. Because it makes fewer assumptions about the relative motion of observers, a broader range of possibilities arise when we move from special to general relativity. For example, one of the axioms of BASAX postulated that the traces of bodies, the paths which bodies make in time, are straight lines. This hypothesis is dropped in general relativity theory, and the traces of bodies may now have quite diverse shapes. In logical terms, this means that the class of models of general relativity theories are larger and more diverse than models of special relativity theory. Mathematically, general relativity theory is more difficult to address; indeed, the language of advanced mathematics is used just to state many of the definitions and describe even the most elementary models. Despite the complications, there is some hope that the logical method which was applied so successfully before may give some insight into general relativity. Indeed, this is the hope that lies behind my current research on acceleration.

4.2. Accelerating from slower to faster than light

Is it possible to accelerate from slower to faster than light? We saw earlier that BASAX implies that an observer can never travel at the speed of light, so, in particular, it is impossible to travel from slower than light to faster than light because at some point the observer must travel exactly at the speed of light. Despite these signs pointing toward the impossibility of the desired acceleration, the statement of the problem does make sense from a kinematic point of view, and we may ask what axioms we must modify in order to allow the possibility of accelerating so much that one can pass a photon. My advisors and I, together with a couple other students, are currently working on this problem. The answer remains unknown, although there are signs that, in fact, it is possible to pass a photon under reasonable, natural relativistic hypotheses.

5. Other activities and lessons learned

In addition to my work on special relativity theory, I have been involved in a number of other activities in Budapest that I would like to share. In October, I attended a logic conference in Linz, Austria, at which I met a number of mathematicians and informed them of my Fulbright work in Hungary. In November, I gave a talk at a conference in honor of the 60th birthday of one of my advisors, István Németi. The audience consisted of mathematicians and logicians who talked out how István's work had influenced their own. My contribution to the conference was to give an introduction to my advisor's work, which remains largely unknown outside of Hungary. My talk was successful and wellattended, the audience asked a number of interesting and intelligent questions, and I met many Hungarian mathematicians. At István's conference I was introduced to the mathematician Mikl´os Ferenczi. Mikl´os has written a large introduction to logic for engineers, and I am assisting him with the translation of his work from Hungarian to English. This term I am also taking two classes through the Budapest Semesters in Mathematics (BSM) program. Two years ago, in the spring of 2001, I was a student in the BSM program while I was an undergraduate student at the University of Minnesota, and it was my first introduction to Hungary. Indeed, I met my advisors while I was studying mathematics here in 2001, and first had the idea of coming back to Hungary to work with them. This term, I am taking Topics in Geometry and Differential Geometry. Both of these classes are related to my work because they focus on geometrical methods used in relativity. Through the same program, I organized a problem-solving seminar. My hope was to propose problems from a famous Hungarian problem book, *Problems and Theorems in Analysis*, by the Hungarian mathematicians György Pólya and Gábor Szegő. This book (and the works of György Pólya in general) has influenced my mathematical development; my goal was to teach in Hungary some of the methods of Pólya and Szegő. Unfortunately, the seminar was not well-attended, although there were a few interested students. Quite recently, a new contact has been made with András Simon of the Alfréd Rényi Institute. András and I may work on problems in logic together.

6. Conclusion

My work has, on the whole, been successful, but it has had its ups and downs. I have learned a number of lessons that I would like to share. First, this has been the first major research project in which I have been involved. I have been given a great deal of freedom to work as I please, and I have gone through a number of times when I have wandered around not quite knowing what to do next. I am grateful for having had this opportunity, because I believe that by facing the struggle of keeping myself engaged in my work I will be a better researcher in the future. Second, the relation between me and my advisors has evolved. In the beginnings of the project, I viewed them as a source for structure. It became clear later on that my advisors did not play this role, and that I was more independent that I had imagined. After an initial orientation period, it saw that it was up to me to propose my own problems. Again, I am grateful for having had an opportunity to struggle with this independence, because I feel that I am a more independent person for having gone through it. Finally, living in Hungary has been a powerful maturing force. I believe that I have grown up considerably by living in Budapest, and I am grateful for the opportunity that the Fulbright Commission has given to me to study here.

References

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