Turán-type Problems

in Extremal Combinatorics

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This paper discusses the research that was done between Gábor Tardos and the author during his stay at the Rényi Institute from September 2003 until May 2004. It presents two major Turán-type results in the field of extremal combinatorics, one in the theory of excluded matrices and the other in topological graph theory.

1. Introduction

Extremal combinatorics has come into its own as a field of research quite recently due to its importance in finite optimization. It is hard to compare processes if one does not know the limitations involved in each, and so the study of these limitations has become crucial in fields such as Operations Research and Computer Science. This paper discusses a class of extremal problems known as Turán type problems, named for Pál Turán, which include problems that aim to maximize a certain valuable while avoiding a given illegal situation.

Section 2 considers a problem concerning the maximum number of 1 entries that can appear in a 0–1 matrix while still avoiding a fixed, smaller matrix. The applications of this problem are contained mostly in Computer Science, where the effectiveness of various search and sort algorithms depend on the number of times it is forced to deal with certain ordered configurations. Apart from this, the result solves one of the most interesting open problems in Enumerative Combinatorics, and so it is important in its own right.

Section 3 studies a situation that arises in many industrial applications. It is often useful to place as many nodes and node-connections onto a region without producing a certain situation. For example, one might hope to squeeze as many interconnected transistors onto a circuit board without having any of the connections cross. Our problem takes a fixed number of nodes and asks how many connections can occur if we wish to avoid a self-intersecting cycle of length 4. This problem has numerous applications in optimization, especially in the decomposition of pseudocircles into pseudosegments (a process that cuts a web of interconnected paths into pieces which maintain a certain intersection property), which could easily be useful in problems concerning ground transportation (where the set of routes forms such a web of paths).

Most importantly, though, each of the solutions we give uses new techniques in combinatorics which will hopefully solve more problems than the two which we will consider here, and so many of the applications have yet to be discovered.

2. A Problem in Excluded Matrices

Extremal graph theory is a well studied field in combinatorial research, but only recently has this research been extended by imposing an order on the vertices. The order, in fact, makes the problem of finding subgraphs substantially harder and so far very few results have appeared in the literature. Zoltán Füredi and Péter Hajnal [4] first considered ordered bipartite graphs by using matrix representation and since then progress has been made by Tardos [10].

The first result presented in this paper is a joint work between Gábor Tardos and examines the extremal problem of how many 1 entries an nXn 0–1 matrix can have that avoids a certain fixed submatrix P. For any permutation matrix P we prove a linear bound, settling a conjecture of Füredi and Hajnal [4]. Due to the work of Martin Klazar [8], this also settles the conjecture of Stanley and Wilf on the number of n-permutations avoiding a fixed permutation and a related conjecture of Alon and Friedgut [1].

2.1 Introduction

This paper settles three related conjectures on avoiding smaller patterns. To state the conjectures we define the term "avoiding" in several contexts.

Definition 1. Let A and P be 0–1 matrices. We say that A *contains* the $k \times l$ matrix P=(p_{ij}) if there exists a $k \times l$ submatrix B=(b_{ij}) of A with b_{ij}=1 whenever p_{ij}=1. Otherwise we say that A *avoids* P. Notice that we can delete rows and columns of A to obtain the submatrix B but we cannot permute the remaining rows and columns. If A contains P we identify the entries of the matrix A corresponding to the entries b_{ij} of B with p_{ij} = 1 and say that these 1 entries of A represent P.

Let $[n] = \{1, 2, \ldots, n\}$. A permutation of $[n]$ is called an n-permutation. We say that an n-permutation σ contains a k-permutation π if there exist integers $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ such that for $1 \leq i, j \leq k$ we have $\sigma\ (x_i)\ \sigma\ < (x_j)$ if and only if $\pi\ (i) < \pi\ (j)$.

Otherwise we say that σ *avoids* π.

The set of finite sequences (words) over $[n]$ is denoted by $[n]^*$. A sequence $a = a_1a_2 \ldots a_1$ $[n]^*$ *contains* the sequence $b = b_1b_2 \ldots b_k \in [m]^*$ if there exist indices $1 \leq x_1 < x_2 < \ldots < x_k \leq l$ such that for $1 \leq i, j \leq k$ we have

Otherwise we say that *a avoids b*.

$a_{x_i} < a_{x_j}$ if and only if $b_i < b_j$.

Note that these notions are intimately related. Clearly the word $\sigma(1)\sigma(2) \ldots \sigma(n)$ will contain the word $\pi(1)\pi(2) \ldots \pi(k)$ if and only if the n-permutation σ contains the k-permutation π, which happens if and only if the permutation matrix of σ contains the permutation matrix of π.

Definition 2. For a 0–1 matrix P let $f(n, P)$ be the maximum number of 1 entries in an $n \times n$ 0–1 matrix avoiding P.

For a permutation π let $S_n(\pi)$ be the number of n-permutations avoiding π. A sequence $a_1a_2 \ldots a_n$ is considered *k-sparse* if $i < j$, $a_i = a_j$ implies $j - i \geq k$. For a sequence b $\in [m]^*$ and $k \geq m$ let $l_k(b, n)$ be the maximum length of a k-sparse word in $[n]$ avoiding b.

The main result of this paper is the following theorem proving the Füredi-Hajnal conjecture (originally posed in [4]).

Theorem 1. ***For all permutation matrices P we have f(n, P) = O(n).***

By a result of Martin Klazar [8] (which we reproduce in Section 2.3) the above result proves the Stanley–Wilf conjecture (stated here as Corollary 2) and also the Alon–Friedgut conjecture (stated here as Corollary 3), which was originally posed in [1]. The Stanley–Wilf conjecture was formulated by Richard Stanley and Herbert Wilf around 1992 but it is hard to find an exact reference. An even earlier source is the PhD thesis of JulianWest [11] of 1990 where he asks about the growth rate of $Sn(\pi)$. His question 3.4.3 is more specific; he asks if $Sn(\pi)$ and $Sn(\pi')$ are asymptotically equal for k-permutations π and π'. Miklós Bóna [2] showed that this conjecture was too strong, however, by finding 4-permutations π and π' with $Sn(\pi)$ and $Sn(\pi')$ displaying different growth rates. Nevertheless, it shows a direct interest in asymptotic enumerations of this kind.

Corollary 2. ***For all permutations there exists a constant $c = c_\pi$ such that $Sn(\pi) \leq c^n$.***

Corollary 3. ***For a k-permutation σ and word $a = \sigma(1)\ \sigma(2) \ldots \sigma(k)$ we have $l_k(a, n) = O(n)$.***

Several special cases of the above conjectures have already been established. Bóna [3] proved the Stanley–Wilf conjecture for *layered* permutations π, that is, for permutations consisting of an arbitrary number of increasing blocks with each of elements of a block smaller than the elements of the previous block. Approximate versions of all conjectures were also established. Using another result of Klazar [5] on generalized Davenport-Schinzel sequences Alon and Friedgut [1] showed approximate versions of their own conjecture and the Stanley–Wilf conjecture where the linear and exponential bounds were replaced by $O(n\gamma(n))$ and $2^{O(n\gamma(n))}$, respectively, with an extremely slow growing function γ related to the inverse Ackermann function. In Section 2.2 we give a surprisingly simple and straightforward proof of Theorem 1. For the reader's convenience we reproduce Klazar's argument on how this result implies the Stanley-Wilf conjecture in Section 2.3. In Section 2.4 we present a multidimensional extension of Theorem 1 in the form of hypergraphs.

2.2 Proof of the Füredi-Hajnal conjecture

Theorem 1 is proved by establishing a linear recursion for $f(n, P)$ in Lemma 7, that in turn is based on three rather simple lemmas. We partition the matrix into blocks. This idea appears in several related papers, e.g. in [8], but we use larger blocks than were previously considered.

Throughout these lemmas, we let P be a fixed kXk permutation matrix and A be an nXn matrix with $f(n, P)$ one entries which avoids P. We assume k^2 divides n. We define to be the square submatrix $S_{ij} = \{a_{i'j'} \in A : i' \in [k^2 i+1, k^2(i+1)], j' \in 2\ [k^2 j+1, k^2(j+1)]\}$.

We let $B = (b_{ij})$ be an (n/k^2) X (n/k^2) 0–1 matrix with $b_{ij} = 0$ if all entries of S_{ij} are zero. We say that a block is *wide* (respectively *tall*) if it contains 1 entries in at least k columns (respectively rows).

Lemma 4. ***B avoids P.***

Proof. Assume not and consider the k 1–entries of B representing P. Choose an arbitrary 1–entry from the k corresponding blocks of A. They represent P, contradicting the fact that A avoids P.

Lemma 5. *Consider the set (column) of blocks $C_j = \{S_{ij} : i = 1 \ldots \frac{n}{k^2}\}$. The number of blocks in C_j that are wide is less than $k\binom{k^2}{k}$*

Proof. Assume not. By the pigeonhole principle, there exist k blocks in C_j that have a 1 in columns $c_1 < c_2 < \ldots < c_k$. Let $S_{d1j}, \ldots, S_{dkj}$ be these blocks with $1 \le d_1 < d_2 < \ldots < d_k \le n/k^2$. For each 1–entry p_{rs}, pick any 1–entry in column c_s of S_{drj}. These entries of A represent P, a contradiction.

Lemma 6. ***Consider the set (row) of blocks $R_i = \{S_{ij} : j = 1 \ldots \frac{n}{k^2}\}$. The number of blocks in R_i that are tall is less than $k\binom{k^2}{k}$***

Proof. The same proof applies as for Lemma 5.

With these tools, the main lemma follows:

Lemma 7. ***For a kX k permutation matrix P and n divisible by k^2 we have***

$$f(n, P) \le (k-1)^2 f\ \frac{n}{k^2}, P + 2k^3\binom{k^2}{k} n:$$

Proof. We consider three types of blocks:

$X_1 = \{$ blocks that are wide $\}$.

$|X_1| \le \frac{n}{k^2} k\binom{k^2}{k}$ by Lemma 5.

$X_2 = \{$blocks that are tall$\}$.

$|X_2| \le \frac{n}{k^2} k\binom{k^2}{k}$ by Lemma 6.

$X_3 = \{$nonempty blocks that are neither wide nor tall$\}$.

$|X_3| \le f(\frac{n}{k^2}, P)$ by Lemma 4.

- X1 = { blocks that are wide }.

This includes all of the nonempty blocks. We bound $f(n, P)$, the number of ones in A by summing estimates of the number of ones in these three categories of blocks. Any block contains at most k^4 *1*–entries and a block of X_3 contains at most $(k-1)^2$ 1–entries.

$$f(n, P) \le k^4|X_1| + k^4|X_2| + (k-1)^2|X_3|$$

$$\le 2k^3\binom{k^2}{k} n + (k-1)^2 f\left(\frac{n}{k^2}, P\right)$$

Solving the above linear recursion gives the following Theorem and also Theorem 1.

We did not optimize for the constant factor here.

Theorem 8. ***For a kXk permutation matrix P we have***

$$f(n, P) \le 2k^4\binom{k^2}{k} n.$$

Proof. We proceed by induction on n. The base cases (when $n \leq k^2$) are trivial. Now assume the hypothesis to be true for all $n < n_0$ and consider the case $n = n_0$. We let n' be the largest integer less than or equal to n which is divisible by k^2. Then by Lemma 7, we have:

$$f(n, P) \leq f(n', P) + 2k^2 n \leq (k-1)^2 f\left(\frac{n'}{k^2}, P\right) + 2k^3 \binom{k^2}{k} n' + 2k^2 n$$

$$\leq (k-1)^2 \left[2k^4 \binom{k^2}{k} \frac{n'}{k^2}\right] + 2k^3 \binom{k^2}{k} n' + 2k^2 n \leq 2k^2 \ (k-1)^2 + k + 1 \ \binom{k^2}{k} n \leq 2k^4 \binom{k^2}{k} n$$

where the last inequality is true for all $k \geq 2$.

2.3 Deduction of the Stanley-Wilf conjecture

For the reader's convenience (and to show the similarities of the two proofs) we sketch here Klazar's argument [8] that the Füredi-Hajnal conjecture implies the Stanley–Wilf conjecture.

Definition 3. For a 0–1 matrix P let $T_n(P)$ be the set of nXn matrices which avoid P.

As we noted in Section 2.1, a permutation σ avoids another permutation π if and only if the permutation matrix corresponding to σ avoids the permutation matrix corresponding to π. Thus if P is the permutation matrix of the permutation π, then $T_n(P)$ contains the permutation matrices of all n-permutations avoiding π. In particular we have $|T_n(p)| \geq S_n(\pi)$.

Assuming the Füredi-Hajnal conjecture Klazar proves the following stronger statement, which in turn implies Corollary 2:

Theorem 9. ***For any permutation matrix P, there exists a constant c depending on P such that*** $|T_n(P)| \leq c^n$.

Proof. Using $f(n, P) = O(n)$ the statement of the theorem follows from the following simple recursion: $|T_{2n}(P)| \leq |T_n(P)| 15^{f(n,P)}$.

To prove the recursion we map $T_{2n}(P)$ to $T_n(P)$ by partitioning any matrix $A \in T_{2n}(P)$ into 2 by 2 blocks and replacing each all-zero block by a 0–entry and all other blocks by 1–entries. As we saw in Lemma 4 the resulting n by n matrix B avoids P. Any matrix $B \in T_n(P)$ is the image of at most 15^w matrices of $T_{2n}(P)$ under this mapping where w is the number of 1 entries in B. Here $w \leq f(n, P)$ so the recursion and the Theorem follow.

The reduction also provides a nice characterization in the theory of excluded matrices:

Corollary 10. ***Let P be a 0–1 matrix. We have*** $\log(|T_n(P)|) = O(n)$ ***if and only if P has at most a single 1 entry in each row and column.***

Proof. The matrices in the characterization are the submatrices of permutation matrices.

For these matrices $\log(|T_n(P)|) = O(n)$ follows from Theorem 9. For other matrices P, $T_n(P)$ contains all of the $n \times n$ permutation matrices (a total of $n!$), so $\log(|T_n(P)|) = \Omega(n \log n)$.

2.4 Multi-dimensional extension

We define the term "avoiding" in one final context.

Definition 4. We denote the set of d-uniform, d-partite hypergraphs $H(V,E)$ with (ordered) partitions $H_1, \ldots, H_d$ each isomorphic to $[n]$ as $\mathcal{H}(n, d)$.

Let $H(V,E) \in \mathcal{H}(n, d)$.

For an edge $\mathbf{e} \in \mathbf{E}$ we define *the projection function* $\pi_t(e) = e \setminus H_t$. We extend this definition so that $\pi_t(H) \in \mathcal{H}(n, d - 1)$ is the hypergraph with vertex set $V \setminus H_t$ and edge set $\{\pi_t(e) : e \in E\}$. Similarly to the extended definition, we define $\pi_t^j(H) \in \mathcal{H}(n, d - 1)$ to be the hypergraph with vertex set $V - H_t$ and edge set $\{\pi_t(e) : e \in E, e \cap H_t = j\}$.

We define $H - H_t \in \mathcal{H}(n, d - 1)$ to be the hypergraph with vertex set $V' = H \setminus H_t$ and edge set $E' = \{e \setminus H_t : e \in H\}$. For an edge $f \in E'$, we define the function $\tau_t(f) = \{e \in E : e \setminus H_t = f\}$. In a way, $\tau_t(f)$ is the set of preimages of the edge f.

We will call a hypergraph $P \in \mathcal{H}(n, d)$ a *permutation hypergraph* if every point is covered by exactly one edge. We will write $\mathcal{P}(k, d) \subset \mathcal{H}(k, d)$ to denote this subset and we define $f(n, k, d) = \max_{P \in P(k,d)} f(n, P)$.

For two hypergraphs $A \in \mathcal{H}(n, d)$ and $B \in \mathcal{H}(k, d)$, we say that A *contains* B if there exists a hypergraph $A' \subseteq A$ which is order isomorphic to B. Notice that we can delete vertices and edges of A to obtain A' but we cannot permute the remaining vertices. If A contains B we identify the edges of A which are order isomorphic to B and say that these edges *represent* B.

Note that these notions reduce to the original theorem for the case $d = 2$. It is also informative to note the similarity to the notions discussed by Klazar in [6] and [7].

Theorem 11.
$f(n, P) = O(n^{d-1})$ for all $P \in \mathcal{P}(k, d)$.

2.5 Proof of the Theorem in Multidimensions

Throughout these lemmas, let $d > 2$ and let $P \in \mathcal{P}(k, d)$ be a fixed permutation hypergraph. Furthermore, we assume that n is a multiple of $m = k^{d/d-1}$ and consider a hypergraph $H(V,E) \in \mathcal{H}(n, d)$ that has $f(n, P)$ edges and that avoids P.

We partition the edges into the following "blocks" :

$$S = \{S_{i_1 \ldots i_d} : i_1, \ldots, i_d \in 0, \ldots, n/m\} = \{e \in E : e \cap H_t \in [mi_t + 1, m(i_t + 1)] \forall t \in [d]\}.$$

Given a block partition of edges S we will construct the *block hypergraph* $B_S(V_S, E_S) \in \mathcal{H}(n/m, d)$ such that $(i_1, \ldots, i_d) \in E_S$ if and only if $S_{i1 \ldots id}$ is non-empty. We define the mapping ▯ : $E_S \rightarrow S$ which maps the edge $(i_1, \ldots, i_d) \in E_S$ to the block $S_{i1 \cdots id}$.

We say that a block S is *t-large* if $|S \cap H_t| \geq k$.

Lemma 12. *B_S avoids P.*

Proof. Assume not and let the edges $b_1, \ldots, b_k$ represent P in B_S. For each b_i, choose an arbitrary edge from $\rho(b_i)$. They represent P, contradicting the fact that H avoids P.

Lemma 13. *Let $q \in [k]$ and $p' \in P - Pq$. Then $P - Pq \in \mathcal{P}(k, d-1)$ and $\tau_q(p')$ is a single edge.*

Hence when one "squashes" P, the new hypergraph is a permutation hypergraph and each new edge is the contraction of exactly one edge in P.

Lemma 14. *Consider the set of blocks $T^j_t = \{S_{i1\ldots id} : i_t = j\}$. The number of blocks in this set which are t-large is less than $f(n/m, k, d-1)\binom{m}{k}$.*

Proof. Assume not. Then for each block S in T^j_t we have $|S \cap H_t| \geq k$ and for any edge in S there are m choices for $e \cap H_t$. So by the pigeonhole principle, there exist a set of blocks $U = \{U_i : i \in 1, \ldots, f(n/m, k, d-1)\}$ and integers $c_1 < c_2 < \ldots < c_k$ such that $\{c_1, \ldots, c_k\} \subseteq U_i \cap H_t$ for all $U_i \in U$.

Now consider the block hypergraph $B_U(V_U, E_U)$. For every block U_i, there is an edge in E_U, so $|E_U| \geq f(n/m, k, d-1)$ and $B_U \subseteq \pi_t(BS) \in \mathcal{H}(n/m, d-1)$. Now let $P' = \pi_t(P) \in \mathcal{P}(k, d-1)$ (by Lemma 13). B_U cannot avoid P' since B_U has more than $f(n/m, k, d-1)$ edges (and by Lemma 12), so let $b_1, \ldots, b_k$ be the edges in E_U that represent the edges $p'_1, \ldots, p'_k \in P'$, respectively and define $S_i = \rho(b_i)$ for $i = 1, \ldots, d$.

Then $S_1, \ldots, S_k \subseteq U$ so $\{c_1, \ldots, c_k\} \subseteq S_i \cap H_t$ for all $i \in 1, \ldots, k$. Also, if one chooses a single edge from each from each of these blocks, these edges represent P'. So for each edge $p'_i \in P'$, let $p_i = \tau_t(p'_i)$ (a single point in P by Lemma 13) and let $y_i = p_i \cap P_t$. Finally, choose edges $e_i \in S_i$ so that $e_i \cap H_t = y_i$.

Clearly the e_i contain the same order as the p_i in all of the partitions except the t-th partition, since $S_1, \ldots, S_k$ were chosen in that manner. Furthermore, the t-th partition has the same order since the edges themselves were chosen from their respective blocks with that property. Hence the e_i represent P in H, a contradiction.

Lemma 15.

$$f(n, k, d) \leq (k-1)^d f\left(\frac{n}{m}, k, d\right) + dnk^d f\left(\frac{n}{m}, k, d-1\right)\binom{m}{k}.$$

Proof. We consider d+1 types of blocks:

- $X_t = \{$ blocks that are t-large $\}$ for $t \in [d]$.

  $|X_t| \leq \frac{n}{m} f(\frac{n}{m}, k, d-1)\binom{m}{k}$ by Lemma 14.

- $X_0 = \{$ blocks that are not t-large for any t $\}$.

  $|X_0| \leq f(\frac{n}{m}, k, d)$ by Lemma 12.

This includes (with some overcounting) all of the nonempty blocks. We bound $f(n, k, d)$, the number of edges in H by summing estimates of the number of edges in these three categories of blocks. Any block contains at most m^d edges and a block of X_0 contains at most $(k-1)^d$.

$$f(n,k,d) \leq (k-1)^d|X_0| + m^d \sum_{t=1}^{d} |X_t| \leq (k-1)^d f(\frac{n}{m},k,d) + dnm^{d-1} f(\frac{n}{m},k,d-1)\binom{m}{k}$$

And since $m^{d-1} = k^d$, the lemma follows. Finding a valid solution for the above linear recursion gives the following Theorem and also Theorem 11. We did not optimize for the constant factor here.

Theorem 16. $f(n, k, d) \leq d^{d+1}\binom{m}{k}^d k^{3d} n^{d-1}$.
Proof. We proceed by induction on n and d.

The base cases in d (when $d = 2$) results from Theorem 11.

The base cases in n (when $n < k^3$) is trivial.

Now assume the hypothesis to be true for all $n < n_0$ and $d < d_0$ and consider the case $n = n_0$, $d = d_0$. We let n' be the smallest integer less than or equal to n which is divisible by $m = k^{d/d-1}$ and write $p = \binom{m}{k}$. Then by Lemma 15, we have:

$$f(n,k,d) \leq f(n',k,d) + dmn^{d-1} \leq (k-1)^d f(\frac{n}{m},k,d) + dpnk^d f(\frac{n}{m},k,d-1) + dmn^{d-1}$$

$$\leq (k-1)^d d^{d+1} p^d k^{2d} n^{d-1} + d(d-1)^d mp^d k^{3d-3} n^{d-1} + dmn^{d-1}$$

$$\leq dn^{d-1}\left[(1-\frac{1}{k})^d d^d p^d k^{3d} + d^d p^d k^{3d-1}\right] \leq d^{d+1} p^d k^{3d} n^{d-1}\left[1 - 1/k + 1/k\right] \leq d^{d+1} p^d k^{3d} n^{d-1}$$

where each inequality is true for $d \geq 2$.

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3. A Problem in Topological Graph Theory

A *topological graph* is a graph without loops or multiple edges drawn in the plane (vertices correspond to distinct points, edges correspond to Jordan curves connecting the corresponding vertices). We assume no edge passes through a vertex other than its endpoint and every two edges have a finite number of common interior points and they properly cross at each (that is, there are no points of tangency). The problem we consider was derived from the following problem, which was first considered by Pinchasi and Radoičić in [9]:

Problem 1. *How many edges can a topological graph of n vertices have such that no* 4 *edges form a self-intersecting cycle?*

They were able to reduce Problem 1 to a purely combinatorial problem regarding cyclically ordered sequences, and this is the problem that we consider.

3.1 Introduction

In this paper we consider *cyclically ordered* sequences of distinct symbols from a finite alphabet. We say that two such sequences are *intersection reverse* if the common elements appear in reversed cyclic order in the two sequences.

For a vertex v of a topological graph G let $L_G(v)$ be the list of its neighbors ordered cyclically according to (the initial segment of) the connecting edge. Pinchasi and Radoičić in [9] noticed the following simple fact:

Fact 1. *If the lists LG(u) and LG(v) are not intersection reverse for the distinct vertices u and v of the topological graph G then G contains a self-crossing cycle of length 4. Moreover, u and v are opposite vertices of a cycle of length 4 in G that has two edges that cross an odd number of times.*

Proof. One has to consider the drawings of the complete bipartite graph $K_{2,3}$ with u and v on one side and three vertices a, b, and c on the other. If it is not intersection-free then

the topological graph contains self-crossing C_4. If it is intersection-free (or every pair of edges cross an even number of times), then $L(u)$ and $L(v)$ has a, b, and c in reverse cyclic order.

Pinchasi and Radoičić used the above fact to bound the number of edges of a topological graph not containing a self-crossing C_4. They showed that such a topological graph on n vertices has $O(n^{8/5})$ edges. Following their footsteps we also use the above property but improve their bound to $O(n^{3/2} \log n)$. This bound is tight apart from the logarithmic factor as one has (abstract) simple graphs on n vertices with $\Omega(n^{3/2})$ edges containing no C_4 subgraph.

In Section 3.2, we discuss the dierences between our proof and the one in [9] and

discuss some of the notions that we will need. Section 3.3 presents a new argument that attempts to take a more visual approach to the problem. Our main technical result is the following, which comes as a result of Theorem 25:

Theorem 17. *Let us be given m cyclically ordered lists of d element subsets of a set of n symbols. If the lists are pairwise intersection reverse, then*

$$d = O\left(\sqrt{n}\log n + \frac{n}{\sqrt{m}}\right).$$

In Section 3.4, we present a result that we did not use in the proof of the bound, but feel is an important observation and may prove useful in further research. All logs are taken to be binary.

3.2 Preliminaries

Much of the proof follows the argument of [9]. We start with an overview of their techniques and comment on similarities and differences in the present proof. Pinchasi and Radoičić break the cyclically ordered lists into linearly ordered blocks. They consider pairs of blocks from separate lists and pairs of symbols contained in both blocks. They distinguish *same pairs* and *different pairs* according the two symbols appearing in the same or in different order. They observe that any pair of symbols that appear in many intervals must produce almost as many same pairs as different pair. On the other hand the intersection reverse property forces that two cyclically ordered lists–unless most of their intersection is concentrated into a single block–contributes much more different than same pairs. Exceptional pairs of cyclically ordered lists are treated separately with techniques of extremal graph theory. They optimize in their choice for the length of the blocks.

We follow almost the same path, but instead of optimizing for block length we consider many block lengths (an exponential sequence) simultaneously. For two intersection reverse lists no block length yields significantly more same pairs than different pairs and we will show that some block length actually gives many more different pairs than same pairs.

Definition 5. We will always use the term *sequence* to denote a linearly ordered list of distinct symbols and *cyclic sequence* to denote a cyclically ordered list of distinct symbols. Clearly, if one breaks up a cyclic sequence into blocks, then the blocks are (linearly ordered) sequences.

A simple case analysis shows that, given cyclic sequences u and v, there is at most one pair of blocks (B_u, B_v) which can contain same ordered pairs.

Definition 6. We define R^u_{uv} as follows: If u has only a single block B, then $R^u_{uv} = B$. Otherwise we define R^u_{uv} to be the unique block in u that has a same ordered pair as some block in v. If $s_{uv} = 0$, then we define $R^u_{uv} = ;\emptyset$.

Now given t sequences, we can construct a t-partite graph $G = \bigcup_{i=1}^{t} A_i$ with each partition A_i having one vertex for every element in the i^{th} sequence. We can then connect two vertices if and only if the elements represented by those vertices are the same. Furthermore, we give the edges a color. Assume (a, a') is an edge from partition A_u to partition A_v. We color (a, a') red if $a \in R^u_{uv}$ and $a' \in R^u_{uv}$ and *blue* otherwise. We will call the set of red edges from A_u to A_v $r(u, v)$ and the total number of red edges R. The entire set of edges will be denoted E.

Definition 7. We define the following quantities:

- $d^{red}(u, v)$ = the number of pairs of red edges (a, a'), (b, b') such that a precedes b in some block in u and b' precedes a' in some block in v. Also
$$D^{red} = \sum_{u \neq v} d^{red}(u, v).$$
- $s^{red}(u, v)$ = the number of pairs of red edges (a, a'), (b, b') such that a precedes b in some block in u and a' precedes b' in some block in v. Also
$$D^{blue} = \sum_{u \neq v} d^{blue}(u, v).$$
- $d^{blue}(u, v)$ = the number of pairs of blue edges (a, a'), (b, b') such that a precedes b in some block in u and b' precedes a' in some block in v. Also
$$S^{red} = \sum_{u \neq v} s^{red}(u, v).$$

This partitions the types of pairs of edges that can come between two blocks. Finally, we define $D = D^{red} + D^{blue}$ and $S = S^{red}$. We would like to establish some results concerning the enumerations of various sets of edges.

Lemma 18. $D^{red} - S^{red} \geq -R/2$.

Proof. Let $B_u = R^u_{uv}$ and $B_v = R^v_{uv}$ and let the set of red edges be ordered $(r_1, r_2, \ldots, r_t) \in B_u$ (where $t = |r(u, v)|$). Then the set of red edges must be ordered $(r_j, \ldots, r_t, r_1, \ldots, r_{j-1}) \in B_v$ for some $j \in [t]$. Hence

$$d^{red}_{uv} - s^{red}_{uv} = \binom{j}{2} + \binom{t-j}{2} - j(t-j) = 2j^2 - 2jt + \binom{t}{2} = \frac{1}{2}(2j-t)^2 - \frac{t}{2}$$

so $d^{red}_{uv} - s^{red}_{uv} \geq -\frac{|r(u,v)|}{2}$. Summing over all pairs $u \neq v$, we get that

$$D^{red} - S^{red} = \sum_{u \neq v} d^{red}(u,v) - s^{red}(u,v) \geq -\frac{1}{2}\sum_{u \neq v} |r(u,v)| = -R/2$$

as required.

The following lemma is trivial, but we use it enough to make mentioning it worthwhile.

Lemma 19. *Assume* $\sum_{i=1}^{n} a_i = A$. Then $\sum_{i=1}^{n} \binom{a_i}{2} \geq \frac{A^2}{2n} - \frac{A}{2}$.

Proof. $\sum_{i=1}^{n} \binom{a_i}{2} = \frac{1}{2}\sum_{i=1}^{n} a_i^2 - \frac{1}{2}\sum_{i=1}^{n} a_i \geq \frac{1}{2}\sum_{i=1}^{n} \left(\frac{A}{n}\right)^2 - \frac{A}{2}$ by Jenson's inequality.

Hence $\sum_{i=1}^{n} \binom{a_i}{2} \geq \frac{A^2}{2n} - \frac{A}{2}$ as required. Finally, we would like to have some information about the total number of edges. Using the size of the alphabet, we can get a nice lower bound on $|E|$.

Lemma 20. *Assume* $d \geq 2\frac{n}{m}$. Then $|E| \geq \frac{m^2 d^2}{4n}$.

Proof. Let a_i be the number of times letter i appears in any word. Then $|E| = \sum_{i=1}^{n} \binom{a_i}{2}$.

But $\Sigma_{i=1}^{n} a_i$ is the total number of letters appearing, which we know to be md, so Lemma 19 implies that

$$|E| \geq \frac{m^2 d^2}{2n} - \frac{md}{2}.$$ Since $d \geq 2\frac{n}{m}$

Since $d \geq 2\frac{n}{m}$, the desired inequality stands.

3.3 The Procedure

We define the following procedure: At step $i = 0$, each circular sequence will have one block (the endpoint can be chosen arbitrarily) so, by definition, all edges will be red. At step $i > 0$, we will divide each of the current blocks into two equal halves (each with size $\frac{d}{2^i}$) and assign the edges the appropriate colors.

Definition 8. We let R_i be the number of red edges at step i and let $\Delta i = R_{i-1} - R_i$.

Since $R_0 = |E|$ and $R_{\log d} = 0$, we have that $\Sigma_{i=0}^{\log d} \Delta_i = |E|$. We let D_i be the value D at step i and similarly S_i the value S at step i.

Lemma 21. $D_i - S_i \leq \frac{md^2}{2^i}$

Proof. Let $k = \frac{d}{2^i}$ (the block size) and let b_{xy} be the number of times x follows y in some block. Then

➡ $\sum_{x<y} (b_{xy} + b_{yx}) = f \frac{m}{k}\binom{k}{2}$ (this counts all pairs)

➡ $D_i = \sum_{x<y} b_{xy} b_{yx}$

➡ $S_i = \sum_{x<y} \binom{b_{xy}}{2} + \binom{b_{yx}}{2}$.

Hence $D_i - S_i = \frac{1}{2}\sum_{x<y} b_{xy} + b_{yx} - (b_{xy} - b_{yx})^2 \le \frac{1}{2} m \frac{d}{k}\binom{k}{2} \le \frac{1}{4} mdk$

Lemma 22. $D_i^{blue} \ge \frac{\Delta_i^2}{8m^2} - \frac{\Delta_i}{2}$.

Proof. At round i, there are D_i blue edges which were red edges in the previous round.

Since they were red, they came from at most m^2 pairs of blocks, so now they come from atmost 4m2 pairs of blocks. Let $\Delta_i(u, v)$ be the number of edges from u to v that changed colors. Then

$$D_i^{blue} \ge \sum_{(u,v)} \binom{\Delta_i(u,v)}{2}$$

so Lemma 19 gives us the required inequality.

Lemma 23. *Let* $d \ge 16\sqrt{n \log n} + 2\frac{n}{m}$

and let $Y \subseteq [\log d]$ such that $\sum_{i\in Y} \Delta_i \ge \frac{|E|}{2}$.

Then $\sum_{i\in Y} \sqrt{D_i^{blue}} \ge \frac{|E|}{16m}$.

Proof. Let $X = \{i : \Delta i \ge 8m^2\}$. Then by Lemma 22, we have that

We would like to show that a lot of edges

$$D_i^{blue} \ge \begin{cases} \frac{\Delta_i^2}{16m^2} & \text{for } i \in Y \cap X \\ 0 & \text{for } i \in Y \setminus X \end{cases}$$

are counted by rounds which fall into the set $Y \cap X$. Using the fact that $|Y \setminus X| \le \log n$, we have

But since

$$\sum_{i\in Y\cap X} \Delta_i = \sum_{i\in Y} \Delta_i - \sum_{i\in Y\setminus X} \Delta_i \ge \frac{|E|}{2} - 8m^2 \log n$$

$|E| \ge \frac{d^2 m^2}{4n}$ and $d^2 \ge 256 n \log n$

we have that $8m^2 \log n \le \frac{|E|}{8}$

Hence $\sum_{i\in Y\cap X} \Delta_i \ge \frac{|E|}{4}$

So placing this into a final sum, we have

$$\sum_{i\in Y} \sqrt{D_i^{blue}} \ge \sum_{i\in Y\cap X} \frac{\Delta_i}{4m} + \sum_{i\in Y\setminus X} 0 \ge \frac{|E|}{16m}.$$

Lemma 24.

$d \geq 16\sqrt{n \log n} + 2\frac{n}{m}$ and let $Y \subseteq [\log d]$

such that $\sum_{i \in Y} \Delta_i \geq \frac{|E|}{2}$

Then $\sum_{i \in Y} 2^{i/2} D_i^{blue} \geq \frac{|E|^2}{2^{10} m^2}$

Proof. By Cauchy-Schwarz,
X

$$\sum_{i \in Y} 2^{i/2} D_i^{blue} \sum_{i \in Y} 2^{-i/2} \geq \left(\sum_{i \in Y} \sqrt{D_i^{blue}} \right)^2.$$

Plugging in Lemma 23 and the fact that

$$\sum_{i \in Y} 2^{-i/2} \leq \sum_{i=0}^{\infty} 2^{-i/2} \leq 4,$$

we have the required inequality.

Finally, we have the tools necessary to prove the promised theorem. We do not optimize for the constant here.

Theorem 25. $d \leq 64\sqrt{n}\log n + 512\frac{n}{\sqrt{m}}$

Proof. If $d \leq 16\sqrt{n \log n} + 2\frac{n}{m}$, the result is trivial, so assume not.

Let $Y = \{i : D_i^{red} - S_i^{red} \geq 0\}$.

We consider two cases:

$\sum_{i \in Y} \Delta_i \geq \frac{|E|}{2}$ Since for each

$i \in Y$, $D_i^{red} - S_i^{red} \geq 0$, we have that $D_i^{blue} \leq D_i^{blue} + D_i^{red} - S_i^{red} =$

D_i - S_i. Using this fact and Lemma 21, we have

$$\sum_{i \in Y} 2^{i/2} D_i^{blue} \leq \sum_{i \in Y} 2^{i/2}(D_i - S_i) \leq \sum_{i \in Y} 2^{-i/2} f m^2$$

Again, $\sum_{i \in Y} 2^{-i/2} \leq 4$,

and by Lemma 24, we have the inequality $\frac{|E|^2}{2^{10} m^2} \leq 4md^2$. Now using Lemma 20, we get that m ≤≥128 $\frac{n}{\sqrt{m}}$

2. $\sum_{i \notin Y} \Delta_i \geq \frac{|E|}{2}$ Since for each

$i \in Y$, $S_i^{red} - D_i^{red} \geq 0$, it makes sense to consider the sum $\sum_{i \in Y} \sqrt{D_i^{blue}} - \sqrt{S_i^{red} - D_i^{red}}$.

When $D_i^{blue} \leq S_i^{red} - D_i^{red}$, $\sqrt{D_i^{blue}} - \sqrt{S_i^{red} - D_i^{red}} \leq 0$.

Otherwise $D_i^{blue} \geq S_i^{red} - D_i^{red}$, so

$$\sqrt{D_i^{blue}} - \sqrt{S_i^{red} - D_i^{red}} \leq \sqrt{}$$

$$\sqrt{D_i^{blue} + D_i^{red} - S_i^{red}} \leq 2^{-i/2}\sqrt{md}$$

by Lemma 21. Hence in both cases, we have

$$\sqrt{D_i^{blue}} - \sqrt{S_i^{red} - D_i^{red}} \leq 2^{-i/2}\sqrt{f m}$$

and therefore

$$\sum_{i \notin Y} \sqrt{D_i^{blue}} - \sqrt{S_i^{red} - D_i^{red}} \leq 4\sqrt{md}$$

Now by Lemma 18,

$$D_i^{red} - S_i^{red} \geq -\frac{R_i}{2} \geq -|E|/2 \text{ so } \sqrt{S_i^{red} - D_i^{red}} \leq \sqrt{|E|}$$

Combining this with Lemma 23 gives the inequality

$$\frac{|E|}{16m} - \log d\sqrt{E} \leq \sum_{i \notin Y} \sqrt{D_i^{blue}} - \sqrt{S_i^{red} - D_i^{red}} \leq 4\sqrt{md}$$

Thus either $\sqrt{|E|} \leq 32m \log n$ or $|E| \leq 64m^{3/2}d$. Plugging in Lemma 20, the first inequality yields $d \leq 2^9 \frac{n}{\sqrt{m}}$.

Since these two cases partition all possibilities, the theorem holds.

3.4 Additional Proof

Here we present an additional observation concerning this problem. It give an upper bound for the number of red edges possible with respect to the block size. When the block size gets lower than √d, this estimate far exceeds the estimate used in the proof of Theorem 25 where we took $R_i \leq |E|$. For larger sized blocks, however, the bound is so bad that it is not worth using. We present it here just in case it turns out to be useful in future investigations into this problem.

Lemma 26. *Let B_u be a block size k and let T_u be a set of circular sequences such that for all $v \in T_u$, $|u \cap v| \geq 2|u|^t$ where $t < 1$. Then $\Sigma_{v2T_u} |v \setminus u| \cap 2|u|^{3-2t}$.*

Proof. Consider the bipartite graph $G=A \cup B$ where A is the set of elements in u and B is the set of sequences in $T_u \cup u$ and where an edge (a, b) exists if and only if sequence a contains element b. We know that G can contain no $K_{3,3}$, but since the vertex corresponding to u is adjacent to all of the vertices in A, we can say that $G' = A \cup B'$ where $B' = T_u$ can contain no $K_{3,2}$. Hence, by counting edges,

$$|B'|\binom{|E|/|B'|}{3} \leq \sum_{v \in B'} \binom{deg(v)}{3} \leq \binom{|A|}{3} \leq \frac{|A|^3}{6}$$

so $|E| \leq 2|u||T_u|2/3$. But we know that $|E| \geq 2|T_u||u|^t$, which implies that $|T_u| \leq |u|^{3-3t}$. Plugging in this new inequality gives $|E| = \Sigma_{v \in T_u} |v \cap u| \leq 2|u|^{3-2t}$ as required.

Lemma 27. *Assume all sequences have a fixed block size of k, and let t < 1. Then $R \leq 4m^{5/3}d^{1/3}k^{2/3}$.*

Proof. Denote by c_i the number of pairs (u, v) such that $|R^u_{uv} \cap v| = i$. Then

$$\sum_{(u,v)} |r(u,v)| = \sum_{i=1}^{2k^t-1} c_i i + \sum_{i=2k^t}^{k} c_i i \leq 2f^2k^t + 2\sum_u \sum_{B_u} k^{3-2t}$$

where the second sum comes from Lemma 26.

But $\sum_u \sum_{B_u} 1 = f\frac{d}{k}$, so the total is at most $2m^2k^t + 2mdk^{2-2t}$.

Now, given k, choose t so that $m^2k^t = 2dmk^{2-2t}$. Then by the AM–GM equality, $m^2k^t + m^2k^t + 2mdk^{2-2t} = 3(2m^5dk^2)^{1/3}$, so $R \leq 4m^{5/3}d^{1/3}k^{2/3}$ as required.

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